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Principle of Event Symmetry

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To accommodate topology change, the symmetry of space-time must be extended from the diffeomorphism group of a manifold to the symmetric group acting on the discrete set of space-time events. This is the principle of *event-symmetric space-time*. I investigate a number of physical toy models with this symmetry to gain some insight into the likely nature of event-symmetric space-time. In the more advanced models the symmetric group is embedded into larger structures such as matrix groups which provide scope to unify space-time symmetry with the internal gauge symmetries of particle physics. I also suggest that the symmetric group of space-time could be related to the symmetric group acting to exchange identical particles, implying a unification of space-time and matter. I end with a definition of a new type of loop symmetry which is important in event-symmetric superstring theory.

1. INTRODUCTION

One of the greatest challenges facing theoretical physics is to understand the structure of space-time at the Planck scale. At such small distances quantum theory and general relativity combine and space-time is replaced by some unknown pregeometry. In the 1960s and 1970s some basic ideas about the small-scale structure of space-time were presented by Finkelstein, Penrose, and Wheeler, but otherwise very little progress was made. In the last decade a growing number of speculative pregeometry models have been studied. At the same time developments in quantum gravity such as string theory and canonical quantum gravity have thrown much light on the microscopic nature of space-time. A bibliography of references on the small scale structure on space-time can be found in my review (Gibbs, 1995c).

One clear message from theories of quantum gravity is that there is a physical minimum distance beyond which the Heisenberg uncertainty princi-

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ple inhibits measurement (Garay, 1994). There are also suggestions resulting from studies of the thermodynamics of black holes that the number of physical degrees of freedom in a volume of space must have a strict finite limit (Bekenstein, 1994). These observations lend credibility to models of discrete space-time, but it is important not to forget the importance of topology. A good pregeometry model may have a dual nature with properties of both discrete and continuous space-time.

The theory of event-symmetric space-time is a discrete approach to quantum gravity in which the exact nature of space-time will only become apparent in the solution. Even the number of space-time dimensions is not set by the formulation and must be a dynamic result. In relation to other pregeometry theories, event-symmetric space-time is closest in spirit to quantum relativity (Finkelstein and Gibbs, 1993) and discrete differential manifolds (Dimakis *et al.*, 1995).

Principles of symmetry are of primary importance in both general relativity and quantum mechanics and might be expected to be of at least as much importance in a combined theory of quantum gravity. However, very few pregeometry models use symmetry in a useful way. Wheeler suggests that symmetry conceals the pregeometric structure and should not be given any importance (Wheeler, 1994). My own belief is that the symmetry so far discovered in nature is just the tip of a very large iceberg. In the event-symmetric approach to pregeometry I take symmetry to be an overriding principle no matter how bizarre the conclusions. Specifically, I argue that space-time symmetry must be enlarged to include invariance under the symmetric group acting on space-time as a discrete set of events. By enlarging the symmetry still further it may be possible to unify space-time symmetry with internal gauge symmetry.

At the present time the best candidate to unify all known forces is superstring theory. The aspects of superstring which are least well understood are its symmetry and geometric foundation. Event-symmetric space-time may be the solution to solving these problems and already there are some interesting models of event-symmetric string theory which will be described in this paper.

Before I go into the details of the theory it may be interesting to recall some of the different philosophical ideas about space-time which have been disputed. As observers we perceive events in our physical environment through our senses. In our minds we possess a model of space and time in which we place these events. Before the 20th century a number of philosophers, notably Mach, suggested that since we do not perceive space and time directly, they should not be regarded as existing absolutely in their own right, but only as a result of relations between material objects. On this basis Mach stated his principle that inertia is determined by all the mass of the universe and is therefore relative to the distant stars. This is a physically testable prediction which is known to be highly accurate.

Many mathematicians, however, took the opposite viewpoint. Space is studied as a geometric object existing independently of matter. Riemann went further, suggesting that matter itself could be just a manifestation of local curvature of space.

Einstein was impressed with Mach's philosophy and hoped that Mach's principle would follow as a consequence of general relativity. Paradoxically he found that Riemann's mathematics of curved geometry was just what he needed to formulate the theory. He showed that the gravitational force was a result of geometrodynamics. The beauty of the result was so persuasive that physicists turned away from the earlier philosophy of Mach toward theories in which matter and the other forces might be understood as a consequence of geometry. Kaluza–Klein theories are the best known of this type, but there are also theories in which particles are thought of as purely geometrical objects such as microscopic black holes or wormholes.

In the light of this it is interesting to look forward to what the event-symmetric physics is going to tell us about the relationship between matter and space-time. The principle is a direct extension of covariance in general relativity with invariance under diffeomorphisms being extended to invariance under any one-to-one mapping. In some of the event-symmetric models I will propose, it is natural to identify the symmetric group acting on space-time with the symmetry under exchange of identical particles. This strongly suggests a return to a Machian point of view in which space-time is seen as a consequence of relationships between matter.

Development of the theory of event-symmetric space-time has been my interest for the last five years and has previously been reported in e-prints available on the internet (Gibbs, 1994a, 1995b). In this paper I include and extend the results of those papers.

2. EVENT-SYMMETRIC SPACE-TIME

General relativity is based on the principle that physics is invariant under any differentiable change of space-time coordinates. To be more precise, general relativity is defined on a manifold M and is invariant under the group of diffeomorphisms on the manifold diff(M). Symmetry principles of this type have become the cornerstone of theoretical physics this century. The recipe for using symmetry in theory cooking goes something like this:

1. Choose a group which you think corresponds to a symmetry of physics.
2. Choose a representation of the group which could correspond to the physical variables.

3. Choose an invariant function of the representation to define the action principle for your theory.

When symmetry is combined with other requirements such as locality and renormalizability in quantum field theory, the constraints on choice are so high that it becomes possible to construct theories with a minimum of empirical input. The idea is so compelling that we might believe the laws of physics are based on some fundamental symmetry principle defined by some universal symmetry group G_U. All known symmetries of physics would be derived from the universal group as residual symmetry left over after spontaneous symmetry breaking. If only we knew what G_U was we would be just a couple of steps away from knowing the laws of physics.

Let us suppose for the moment that this is really true. What could we say about the group G_U? It must contain a subgroup isomorphic to the symmetry of general relativity,

$$\mathrm{diff}(M) \subset G_U$$

and another isomorphic to the gauge group,

$$G^M \subset G_U$$

This immediately raises a question: Is the topology of the manifold M determined by the universal group? The diffeomorphism groups on two different manifolds are not isomorphic if they have different topologies. For aesthetic reasons we might prefer that the topology of the space-time manifold is not written into the laws of physics since it would fix the global properties of the universe. There are also arguments from microscopic physics for the same conclusion. Wheeler first pointed out that in a model of quantum geometrodynamics the fluctuations of space-time at the Planck scale would be so great that space-time would be reduced to a foam of virtual wormholes (Wheeler, 1957). The topology of space must be continually changing and quantum gravity must include a sum over all possible space-time topologies. The arguments in favor of topology change have only become stronger with time (Balachandran *et al.*, 1995).

This forces us to conclude that the universal group must contain the diffeomorphism groups for an infinite number of topologically different manifolds. The puzzle that this presents was discussed by Witten when trying to reason what would be the universal group of string theory (Witten, 1993). We must find a group that contains all the allowed diffeomorphism groups. One possibility might be to simply take the direct product of all the groups, but this would define a universe made up of many independent manifolds, which is not what we would want.

Witten's puzzle seems to epitomize the incompatibility between general relativity and quantum mechanics. There may be many ways of resolving it, including the possibility of giving up the fundamental role of symmetry or discarding topology change. In this paper we explore another simple but radical solution. The diffeomorphism group on a manifold is a subgroup of all one-to-one mappings on the manifold, otherwise known as the symmetric group on the set of events in the manifold. The symmetric group is independent of the topological structure of the manifold and is therefore isomorphic to the symmetric group on any other manifold or any other set which has the same cardinality,

$$\text{diff}(M) \subset S(M) \cong S(\aleph_1)$$

It follows that the diffeomorphism group for any manifold whatsoever is isomorphic to a subgroup of the symmetric group and therefore, if the universal group contains the symmetric group acting on space-time events, then topology change is possible. This simple observation leads to the following definition.

Definition. A model of space-time is said to be *event-symmetric* if it is invariant under the symmetric group acting on space-time events, or a larger group which has a homomorphism onto the symmetric group,

$$S(E) \subset G_U/K$$

To satisfy this definition it is not necessary to have an uncountable number of space-time events. A model with the symmetric group $S(\aleph_0)$ would be event-symmetric. It is convenient to regularize the number of space-time events to a finite number N and take the large-N limit while scaling some of the parameters of the model as functions of N. This approach is valid since a manifold with an uncountable number of events can be densely covered with a countable number of events.

An important example of a group with a homomorphism onto the symmetric group is the braided group $B(N)$. The universal symmetry G_U is likely to be a larger structure such as a matrix group like $U(N)$ which contains the symmetric group $S(N)$ represented by permutation matrices. Corresponding symmetry structures for the braid group would be the quantum matrix groups. As we shall see, the principle of event-symmetric space-time becomes more and more interesting as we seek to extend the symmetric group to the most general symmetry possible.

3. SIMPLE MODELS

In event-symmetric space-time there is no continuous time parameter. This should be an advantage in physical situations where time might break down, i.e., at singularities. On the other hand, it makes it unclear how to define quantum models. The simplest way to proceed is to start from a path integral approach and generalize.

The construction of the most general quantum system needed for our purposes is as follows:

- Define a system of field variables $F = (\varphi_1, \ldots, \varphi_n)$. Each one may be real, discrete, or a Grassmann anticommuting variable.
- Define an action functional on the field variables $S(F)$.
- Calculate the partition function

$$Z = \int e^{iS}\, d^n\varphi$$

- Define observables as functionals on the variables $O_i(F)$.
- Calculate expectation values of the observables.

$$\langle O \rangle = \frac{\int O e^{iS}\, d^n\varphi}{Z}$$

- Finally, it may be necessary to take a limit of some sequence of such models in which $n \to \infty$.

For such a model to make sense as a quantum system it is necessary that the action functional S is real and the integral well defined. It is also of interest to study models where S is imaginary. In that case we can write

$$S = i\beta E$$

$$Z = e^{-\beta E}$$

$$\beta = \frac{1}{kT}$$

so such a model can be interpreted as a classical statistical physics system at a temperature T. It is common practice in numerical lattice theory to replace a quantum system with a statistical one obtained by performing a Wick rotation from the Lorentzian sector to the Euclidean sector. In lattice quantum gravity it is also possible to replace Einstein gravity by a statistical model which can be regarded as gravity in a Riemannian sector. It is not yet known how valid such a transformation is, but it is certainly worth studying.

There are also both quantum and statistical models of event-symmetric systems. Ultimately we must be interested in quantum systems, but it is possible to gain much insight into the nature of event-symmetric space-time by studying toy models, most of which are statistical in nature.

To illustrate this we shall solve the event-symmetric Ising model. This consists of a large number N of ferromagnets represented by spin variables

$$s_a = \pm 1, \qquad a = 1, \dots, N$$

Each spin interacts equally with every other spin according to the energy function

$$E = \sum_{a<b} s_a s_b$$

This has $S(N)$ invariance since it is symmetric under spin permutations. It has an additional Z_2 invariance under global spin reversal. Solving the partition function of this model is not very difficult,

$$Z = \sum_{\{s_a\}} e^{-\beta E}$$

Write this as a sum over K negative spins and $N - K$ positive spins,

$$Z = \sum_{K=0}^{N} \binom{N}{K} \exp\left\{\frac{\beta}{N}\left[\frac{N}{2}(N-1) - 2K(N-K)\right]\right\}$$

In the large-N limit this can be approximated (up to a constant factor independent of N) by an integral over a variable

$$p = K/N$$

$$Z \propto \int_{p=0}^{1} dp \exp(N\{\bar{\beta}[\tfrac{1}{2} - 2p(1-p) - p\ln(p) - (1-p)\ln(1-p)\})$$

In this equation we have scaled β as a function of N such that

$$\bar{\beta} = \beta/N$$

is kept constant as $N \to \infty$.

The function in the exponential has one minimum at $p = 1/2$ for $\beta < 1$ and two minima for $\beta > 1$. The large-N limit forces the system into these minima, so there is a phase transition at $\beta = 1$ with the Z_2 symmetry broken above.

Also of some interest is the gauged Ising model in which the spin variables are placed on event links,

$$s_{ab} = \pm 1, \qquad a < b$$

The energy is now a sum over triangles formed from three links

$$E = \sum_{a<b<c} s_{ab} s_{bc} s_{ac}$$

This model again has an $S(N)$ event-symmetry and the Z_2 symmetry is now extended to a gauge symmetry. This is already too complicated to solve exactly by any obvious means.

4. HIDDEN SYMMETRY AND MOLECULAR MODELS

It will be difficult to accept the principle of event-symmetric space-time without a correspondence principle which reduces an event-symmetric model to recognized theories of physics. In particular it will be necessary to explain how the symmetric group is reduced to the diffeomorphism group on a (3 + 1)-dimensional manifold which we know as the invariance of general relativity.

An obvious possibility is that there may be a mechanism of spontaneous symmetry breaking which breaks the symmetric group and leaves the diffeomorphism group as its residual symmetry. By analogy with such mechanisms in statistical mechanics and particle physics we might suppose that there are phase transitions at high energy scales above which the event-symmetry is restored. This is difficult to imagine, but fortunately nature has provided us with a familiar phenomena which, by analogy, can give us an intuitive feel for how such a mechanism might operate, namely, soap film bubbles!

Consider the way in which soap bubbles could arise in a statistical physics model of molecular forces. The forces should be functions of the relative position vectors $\mathbf{X}_a$ and orientation vectors $\mathbf{U}_a$ of N soap molecules. For simplicity kinetic energy is neglected and a potential energy function will be defined,

$$E = \sum_{ab} V(X_a, X_b, U_a, U_b)$$

A partition function is then derived,

$$Z = \int e^{-\beta E}\, dX\, dU$$

The potential should tend rapidly to a constant at large distances in order to suppress long-range interactions, and should be invariant under global translations and rotations. Furthermore, the potential should be invariant under exchange of any two molecules. This introduces a symmetry described by the symmetric group $S(N)$. An analogy then exists between the molecular model and a model of event-symmetric space-time. Molecules correspond to space-time events.

The statistical behavior of the model will depend on the form of the potential energy function. It must be chosen very carefully for there to be a phase in which bubbles form. The forces must favor alignment of the molecules in such a way that they tend to form two-dimensional surfaces at the minimum energy.

The distance between each pair of molecules is given by

$$r_{ab} = |X_a - X_b|$$

and the angle between the orientation of a molecule and the line joining it to another is

$$\cos(\theta_{ab}) = \frac{U_a \cdot (X_a - X_b)}{r_{ab}}$$

A suitable potential is

$$V = [4r_{ab}^{-1} - \sin^2(\theta_{ab}) - \sin^2(\theta_{ba})]e^{-r_{ab}}$$

In the case of a system with just three molecules the minimum configuration is an equilateral triangle with length of side $r = 1 + \sqrt{3}$ and each molecule orientated perpendicular to the triangle. Many molecules will likewise try to arrange themselves in triangles which will join to a planar lattice. The attraction of molecules will draw the molecules closer together to a spacing of $r = 2.13\ldots$. The attraction at long distance is too weak to destabilize this configuration.

At zero temperature the molecules will fall into the low-energy two-dimensional lattice. If there is a large but finite number of molecules, they will almost certainly arrange themselves on the surface of a polyhedral structure which would appear like a frozen crystalline bubble. We are more interested in what will happen at nonzero temperature. It is impossible to be certain of the behavior without detailed analysis or a numerical simulation, but for the purposes of this example it is enough to conjecture.

At low temperatures the bubble will start to melt and it is easy to imagine that it will start to deform in shape. It is likely that there will be a low temperature at which there is a phase transition. Above this temperature the molecules will no longer stay in the lattice formation, but will be able to flow around the bubble. This melting phase transition can be compared to a model of space-time as a critical solid (Orland, 1993). It will be possible for the bubble to change topology by splitting or forming holes. At a higher temperature the bubble must eventually evaporate to form a gas with no apparent topological form.

The conjecture, therefore, is that the bubble model has three phases as temperature changes, a solid phase, a liquid phase, and a gas phase. The interpretation in terms of event-symmetric space-time is that the liquid bubble phase is analogous to two-dimensional quantum gravity in its Riemannian sector. At high temperatures space-time evaporates into a gas of events. In the gas phase the event-symmetric nature of space-time is evident, but time and space as we know them have no meaning. The dimension of space-time has changed at the phase transition from two to three.

In the liquid phase space-time appears to have recognizable properties such as curvature and its event-symmetric nature is no longer evident. We might say that the symmetry of event-symmetric space-time has been spontaneously broken leaving diffeomorphism invariance as a residual symmetry, but some caution is needed. There is no apparent order parameter which would enable us to distinguish qualitatively between the liquid and gas phases. Furthermore, the bubbles can change topology, so we cannot identify the diffeomorphism group of one specific manifold as the residual symmetry. The model actually has a more general phase diagram in which density is a parameter as well as temperature. The density can be controlled by placing the molecules in a finite-sized box. It is well known that in the phase diagram of water it is possible to go from the gas phase to the liquid phase without passing through a phase transition if a high pressure is applied. The same thing may happen with the bubble model.

In view of this it is preferable to say that event-symmetry is hidden rather than broken. It is worth recalling that in general relativity diffeomorphism invariance is also hidden without being broken. There is no evidence of space-time curvature at human distance scales and before the theory of general relativity it was not at all obvious that physics was invariant under general changes of coordinate system beyond the Poincaré transformations. Similarly I propose that physics is invariant under permutations of space-time events even though it does not appear to be the case.

The physical interpretation of the gas phase is spectacular. Space-time itself may evaporate at very high temperature or density, with changes of space-time dimension or possibly loss of all concept of dimension. If the principle of event-symmetric space-time holds, then this must be the fate of matter when it is compressed at the singularity of a black hole. A similar description of the initial state of the universe may be possible.

It is probable that both the bubble model and real physics have a richer phase diagram in the high-density and high-temperature corner than that described here.

5. TOPOLOGY AND RANDOM GRAPHS

The molecular models of the previous section require an external space in which to embed bubbles representing space-time. One of the strengths of the theory of general relativity is that it formulates curved space-time intrinsically without the need to refer to any external space. Most of the event-symmetric models are also intrinsic in nature, but the lack of an external space-time makes it more difficult to see how a finite-dimensional space-time could arise through a mechanism of symmetry hiding.

The simplest type of model for which this might be possible are random graphs in which N nodes, or space-time events, are randomly pairwise connected by $\frac{1}{2}N(N - 1)$ links. Each graph is defined by link variables l_{ab}, $a < b$, which are conventionally give the value 1 if the nodes a and b are linked and 0 otherwise. Such systems have occasionally been studied as pregeometric models of space-time (Dadic and Pisk, 1979; Antonsen, 1994; Requardt, 1995).

An event-symmetric action (or energy) for a random graph is a function of the graph which is invariant under permutations of the nodes. For example, actions defined as functions of the total number of links L and the total number of triangles T in the graph would be event-symmetric. The partition function might be defined as follows:

$$L = \sum_{a<b} l_{ab}$$

$$T = \sum_{a<b<c} l_{ab} l_{bc} l_{ac}$$

$$E = T - \alpha L$$

$$Z = \sum_{\{l_{ab}\}} e^{-\beta E}$$

It is interesting to see if we can define dimensionality on a random graph. For a given node we can define a function $L(s)$, the number of nodes which can be reached by taking at most s steps along links. If $L(s)$ obeys a power law on an infinite graph for all nodes,

$$l(s) \to s^D \qquad \text{as} \quad s \to \infty$$

then the graph has dimension D. There are other ways to define dimensionality, including at least one which works for finite graphs (Evako, 1994).

For the example partition function above we might hope that there is a phase in which the expectation value of dimension takes some interesting value like 3 or 4. The action favors triangles in the graph, while disfavoring links. If the balance between the two were to favor structures of low dimension, then we would have a similar mechanism of space-time formation and event-symmetry hiding as we did in the soap-film model, but in this case there would be no artificial extrinsic space in which it was embedded.

In fact it seems to be quite difficult to construct random graph models which dynamically generate space-time in a fashion similar to the soap-film model. There is at least one model which manages to easily produce a one-dimensional space-time. The action is defined as

$$V_a = \sum_{b>a} l_{ab} + \sum_{b<a} l_{ba}$$

$$E = \sum_{a} (V_a - 2)^2$$

V_a is the valence of event a and the energy function will be minimized when there are exactly two links connected at each node. This will obviously result in linear structures at low temperatures.

Random graph models can be studied in detail either analytically using such methods as mean-field theory, or numerically using Monte Carlo algorithms. Through careful analysis, it may be possible to contrive an action which generates manifolds of two, three, or four dimensions. Here I will choose to skip past those avenues, which are likely to be dead ends, and follow another which seems to lead to better things.

The concept of random graph can be extended by introducing higher dimensional variables. A variable similar to the link variable but with three event indices, t_{abc}, $a < b < c$, could indicate the triple connection of the vertices of a triangle if its value is one and the absence of a connection if its value is zero. It is convenient to extend the array of values using antisymmetry and allow its elements to take on values -1, 0, or 1,

$$t_{abc} = -t_{bac} = -t_{acb}$$

With these variables it is possible to construct actions which force the triangles to join together forming two-dimensional surfaces at low temperature just as it is possible to form one-dimensional structures with random graphs. For example, if

$$L_{ab} = \sum_{c} t_{abc}$$

$$E_1 = \sum_{ab} L_{ab}^2$$

$$E_2 = \sum_{abc} t_{abc}^2$$

The term E_1 will be minimized when an equal number of positive and negative variables meet at each edge. This corresponds to an orientated triangulated surface which is allowed to cross itself. Other terms such as E_2 can be included in the action to control the area of the surface.

In this way it is possible to define systems which are event-symmetric but which also approximate dynamical triangulations of surfaces as used with considerable success in numerical studies of two-dimensional Riemannian quantum gravity (Boulatov *et al.*, 1986). An important aspect of such a system is that it automatically includes a sum over different surface topologies. Obviously the principle can be extended to variables of dimension higher than two, by straightforward generalization to antisymmetric forms with more indices corresponding to tetrahedrons and higher order simplices in the graph.

There are two important lessons to be learnt here. The first is that higher dimensional variables are likely to give more interesting models than those which just use site and link variables. The second is that systems incorporating a suitably weighted sum over topologies can be considered event-symmetric. As a topic for future research it would be worthwhile to consider what constraints event-symmetry imposes on the weightings in such a sum.

One displeasing aspect of both the random graph models and the molecular models is that the number of dimensions of space-time which they form is put in artificially. Ideally we would like to see the number of dimensions arise as a purely dynamical result. Perhaps the number of dimensions should be able to change through phase transitions. This suggests we should consider models with a mixture of variables of different dimensions. An elegant model might include the link and triangle variables defined above along with variables corresponding to simplices of all other possible dimensions,

$$s, \quad v_a, \quad l_{ab}, \quad t_{abc}, \ldots$$

If each variable is antisymmetric in all indices and there are N events, then the sequence will stop with a variable of N indices. I will not endeavor to consider what might be suitable terms to use in an action with such variables since new principles would be needed to find them. At this point I just want to note the fact that the total number of variables is 2^N. This is a huge number in comparison to the event-symmetric Ising model, which has N variables and 2^N states.

6. GAUGE SYMMETRY AND MATRIX MODELS

The random graph models and their generalizations use variables which can take on one of a number of discrete values. Such models allow us to incorporate event-symmetric space-time which we propose as an extension of the diffeomorphism invariance of general relativity. In particle physics we are familiar with other symmetries represented by continuous Lie groups. It is conceivable that such symmetries could emerge in a discrete model in some limit, but the philosophy behind event-symmetric space-time dictates that symmetries should appear exactly in the most fundamental formulation. Furthermore, it would be pleasing if the space-time symmetries could be unified with the internal gauge symmetries of particle physics.

For this reason I prefer to consider models with continuous rather than discrete variables. We might also remark that if fermions and supersymmetry are to be included, we will also have to permit anticommuting Grassmann variables. While the discrete-variable models have the character of mathematical logic, graph theory, and combinatorics, models with continuous variables will naturally have the character of algebraic mathematics.

Just as almost any physical continuum model can be discretized to produce a lattice theory, it is also possible to produce event-symmetric models corresponding to scalar field theories and gauge theories. The Wilson formulation of lattice gauge theory (Wilson, 1974) can be immediately given an event-symmetric counterpart in which the cubic lattice is replaced with a graph of N events in which each one is linked to each other and a matrix group variable is assigned to each link. Gauge-invariant actions can be defined in terms of the sum over the trace of products taken around each triangle in the graph.

While such models may be of some interest in other contexts (Rossi and Tan, 1995), they fail to satisfy our needs here because, first, there is no mechanism which allows the links to connect to form different topologies, and second, the symmetric group is not unified with the gauge group. The first defect may be remedied by combining a random graph model with a gauge model to form a kind of gauge glass (Bennet *et al.*, 1987), but to cure the second we must go further.

Consider event-symmetric models in which we place real-valued field variables A_{ab} on links joining all pairs of events (a, b). Such models are analogues of the random graph models with the discrete variables replaced by continuous ones. A suitable action must be a real scalar function of these variables which is invariant under exchange of any two events.

The link variables A_{ab} can be regarded as the elements of a square matrix A. If the direction of the links is irrelevant, then the matrix can be conveniently taken to be either symmetric or antisymmetric. If there are no self-links, the diagonal terms are zero, so it is natural to make the matrix antisymmetric,

$$A_{ab} = -A_{ba}$$

A possible four-link loop action is

$$\begin{aligned} S &= \beta \sum_{a,b,c,d} A_{ab}A_{bc}A_{cd}A_{da} + m \sum_{a,b} A_{ab}^2 \\ &= \beta \operatorname{Tr}(A^4) + m \operatorname{Tr}(A^2) \end{aligned}$$

This action is invariant not only under the symmetric group acting on events, but also the orthogonal group acting as similarity transformations on the matrix. The symmetric group $S(N)$ is incorporated as a subgroup of $O(N)$ represented by matrices with a single one in each row or column and all other elements zero, in such a way that the matrix permutes the elements of any vector it multiplies.

This is an appealing scheme since it naturally unifies the $S(N)$ symmetry, which we regard as an extension of diffeomorphism invariance, with gauge symmetries. If the symmetry broke in some miraculous fashion, then it is conceivable that the residual symmetry could describe quantized gauge fields on a quantized geometry.

Consider, for example, a discrete gauge $SO(10)$ symmetry on a four-dimensional periodic hypercubic lattice of $L = M^4$ points. The full lattice gauge symmetry group $Lat(SO(10), M)$ is generated by the gauge group $SO(10)^L$ and the lattice translation and rotation operators. A matrix representation of this group in $10L \times 10L$ orthogonal matrices can be constructed from the action of the group on a ten-component scalar field situated on lattice points. The lattice group is therefore isomorphic to a subgroup of an orthogonal group.

$$Lat(SO(10), M) \subset O(10L)$$

We can imagine a mechanism by which the $O(10L)$ symmetry of a matrix model broke to leave a residual $Lat(SO(10), M)$ symmetry. It seems highly unlikely, however, that such an exact form of spontaneous symmetry breaking could arise naturally.

Random matrix models have been extensively studied in the context where N is interpreted as the number of colors or flavors. The event-symmetric paradigm suggests an alternative interpretation in which N is the number of space-time events times the number of colors. This interpretation has been considered before (Kaplunovsky and Weinstein, 1985).

This suggestion for unification of space-time and internal gauge symmetry might be compared with the similar achievement of Kaluza–Klein theories where space-time is extended to have more dimensions and the symmetry is broken by compactification of one or more of the dimensions. With matrix models the symmetry is much larger and could be compared with a Kaluza–Klein theory which had an extra dimension for each field variable (Kaneko and Sugawara, 1983).

An interesting result for matrix models which is responsible for them attracting so much attention is that the perturbation theory of a matrix model in a large-N double-scaling limit is equivalent to two-dimensional gravity or a $c = 0$ string theory ('t Hooft, 1974; Kazakov, 1989; Fukuma *et al.*, 1994).

We have discussed matrix models with an $O(N)$ symmetry, but models based on Hermitian matrices and having unitary $U(N)$ symmetry are equally interesting, as are models with invariance under the symplectic groups $Sp(N)$. It is just as easy to construct supersymmetric matrix models using the familiar families of supersymmetry matrix groups $U(L|K)$ and $OSp(L|K)$ (Gilbert and Perry, 1991; Alvarez-Gaume and Manes, 1991; Yost, 1992).

As an example we might use super-Hermitian matrices which take a block form as follows:

$$S = \begin{pmatrix} A & B \\ iB' & C \end{pmatrix}$$

where A is a Hermitian $K \times K$ matrix of commuting variables, B is a $K \times L$ matrix of anticommuting variables, and C is a Hermitian $L \times L$ matrix of commuting variables. The supertrace is defined as

$$s\,\mathrm{Tr}(S) = \mathrm{Tr}(A) - \mathrm{Tr}(C)$$

Actions defined with terms expressed as the supertrace of powers of the supermatrices are invariant under a $U(K|L)$ supersymmetry. This can be interpreted as an event-symmetric model with two types of event since the supergroup has a subgroup isomorphic to $S(K) \otimes S(L)$.

7. LOCALITY AND TENSOR MODELS

Just as random graph models can be generalized to models with higher dimensional variables, matrix models can likewise be generalized to tensor models. The action can be a function of any set of scalars derived from the tensors by contraction over indices, with the indices ranging over space-time events. Such models have the same $O(N)$ symmetry as matrix models.

In tensor models it is often useful to associate tensors which have certain symmetry constraints with geometric objects having the same symmetry in such a way that the indices correspond to vertices of the object. For example, a rank 3 tensor which is symmetric under cyclic permutations of indices,

$$T_{abc} = T_{bca}$$

can be associated with a triangle joining the three vertices a, b, and c. If, in addition, the tensor is made fully antisymmetric, then degenerate triangles with two or more vertices at the same event are eliminated and the sign change is useful to indicate orientation reversal of the triangle. Often models of interest use antisymmetric rank-d tensors which can be associated with a system of orientable d-simplices.

We should look for a tensor model with a symmetry-hiding mechanism such that the dynamics separates some events which can then be regarded as being at far distances on a manifold, while others remain close to each other. In other words, we need to generate local interaction. Event-symmetric space-time seems to be contrary to locality, but happily there are principles of locality which can be invoked independently of any event-symmetry-hiding mechanism.

In each of the models we have looked at there are field variables which have an association with one or more events. In matrix models the matrix element A_{ab} is associated with two events indexed by a and b. They represent an amplitude for the connection of those two events as linked neighbors in space-time. In tensor models a tensor of rank r is likewise associated with r events. When symmetry hiding occurs we expect the events to somehow spread themselves over a manifold. A field variable associated with events which are not near neighbors should be physically insignificant; this will usually mean that it is very small. Field variables which are associated with a local cluster of events can be large and would be significant in a continuum limit. Two such variables which are localized around different parts of the manifold should not be strongly correlated. They must therefore not appear in the same interaction term of the action unless multiplied by some small field variable.

This heuristic picture leads to a definition of locality in which interaction terms in the action are excluded if they factor into the product of two parts which do not share events. For example, in a two-matrix model with matrices A and B the action could contain terms such as $\mathrm{Tr}(ABAB)$ but not $\mathrm{Tr}(AB)^2$ or $\mathrm{Tr}(A)\mathrm{Tr}(B)$.

More precisely, we can define an *interaction graph* corresponding to any interaction term. The graph would have a node for each component variable in the term. Two nodes are then linked if the variables are associated with at least one event in common.

We then say that the model satisfies the *weak locality principle* if all interaction graphs are connected. We will also say that it satisfies the *strong locality principle* if every pair of nodes is linked in all interaction graphs, i.e., they are triangles, tetrahedrons, or higher dimensional simplices.

As an example, a matrix model with terms given by the traces of powers of the matrix

$$I_n = \mathrm{Tr}(A^n)$$

is weakly local because the interaction graphs are at least n-sided polygons. If the model includes only terms up to I_3, then it is strongly local.

It is reasonable to expect that physical event-symmetric field theories would have to be at least weakly local since otherwise nonlocal interactions would persist after a symmetry-hiding mechanism has taken effect. There seems to be no special reason to demand that a theory should be strongly local, but it is notable that this condition often reduces the number of possible interaction terms from infinity down to a few without seeming to exclude the most interesting models.

There is one particular form of tensor model which deserves a brief mention here. It is defined with simplex variables such as the antisymmetric rank 3 tensor T_{abc} associated with triangles. We define an action with terms whose connectivity represents a simplex of one higher dimension, e.g.,

$$S = \sum_{a,b,c,d,e,f} T_{abc} T_{ade} T_{bdf} T_{cef}$$

Just as the perturbation theory of a matrix model describes randomly triangulated surfaces, the perturbation of these tensor models defines random simplicial models of higher dimensional surfaces (Ambjorn *et al.*, 1991; Sasakura, 1991). These tensor models do not exhibit the same universality properties which make the matrix models so powerful. This fault has been corrected by Boulatov, who replaces tensors with multivariate functions on groups (or quantum groups) and defines an action which generates three-dimensional topological lattice field theory (Boulatov, 1992).

8. PARTICLE MODELS AND CLIFFORD ALGEBRAS

We have seen how antisymmetric tensor forms can be associated with simplices in event-symmetric space-time and how they might interact together to form manifolds. We will now explore the possibility of a model which includes such variables on simplices of all possible dimension, i.e., the model is defined by a sequence of antisymmetric forms,

$$\xi, \quad \xi_a, \quad \xi_{ab}, \quad \xi_{abc}, \ldots$$

Since there are only a finite number N of events, the family will end with a rank N tensor having only one independent component.

There are many actions which could be constructed from these tensors if we just require the $O(N)$ symmetry. Such models have a huge number of degrees of freedom, one for each possible simplex with vertices on space-time events. Perhaps we could impose a much larger symmetry so as to reduce the number of possible models and at the same time the effective number of degrees of freedom.

A natural way forward is to interpret the family of antisymmetric forms as the components of either an exterior algebra or a Clifford algebra. Here we choose the latter option. A set of gamma operators form the generators of the algebra modulo the usual anticommutator relations.

$$\gamma_a, \quad a = 1, \ldots, N$$

$$[\gamma_a, \gamma_b]_+ = 2\delta_{ab}$$

It follows that the algebra has dimension 2^N and an element can be written

$$\Xi = \xi + \sum_a \xi_a \gamma_a + \sum_{a,b} \xi_{ab} \gamma_a \gamma_b + \cdots$$

A Clifford algebra is an associative algebra with unit and it has a Z_2 grading given by the parity of the number of gamma operators in a product. The graded commutator is therefore a product for a Lie superalgebra. This supersymmetry is much larger than the $O(N)$ symmetry of the general tensor model and from now on we will impose it as a symmetry of our models. It is well known that the second-order operators $\gamma_a \gamma_b$ generate the orthogonal Lie algebra, so event-symmetry is contained within this algebra.

Clifford algebras play several useful roles in particle physics. For example, they are of crucial importance in construction of spinors and supersymmetry. These points in themselves are sufficient to justify their use here. However, there is a third role played by Clifford algebras which may be even more significant. The single gamma matrices together with the unit generate a Lie superalgebra which is known as a Heisenberg algebra. If N is even, the operators can be paired to form a system of $N/2$ fermionic creation and annihilation operators,

$$b_i = \tfrac{1}{2}(\gamma_{2i-1} + i\gamma_{2i})$$
$$b_i^{\dagger} = \tfrac{1}{2}(\gamma_{2i-1} - i\gamma_{2i})$$

From this we deduce that the Clifford algebra is isomorphic to the algebra of fermionic operators and is effectively a Fock space for a species of identical fermions and their antiparticles. The importance of this is that it links the event-symmetry of space-time to the symmetry of identical particle exchange and suggests a realization of Mach's claim that space-time is generated by interactions of matter.

To construct an event-symmetric model we treat the components of the algebra as field variables. Because of the supersymmetry it is necessary to take the odd-rank tensors as anticommuting Grassmann variables. We must define an action which is an invariant of the supersymmetry. The highest rank operator of the algebra is usually written

$$\gamma_{N+1} = \prod_a \gamma_a$$

which has a pseudoscalar component ξ^*. We discover that the linear function I_1 mapping the algebra onto this component is an invariant,

$$I_1(\Xi) = \xi^* \Rightarrow I_1([\Xi, \Delta]) \equiv 0$$

An infinite sequence of invariants can be generated by applying this function to powers,

$$I_n(\Xi) = I_1(\Xi^n)$$

If these are to be suitable terms in an action functional, then N must be even, otherwise the invariants are anticommuting variables. Examining the form of these invariants reveals a dramatic locality problem. Whereas we wished all terms to be formed from local contractions over indices, we find that each term has products of tensor components which include every index exactly once. This problem is resolved by observing that a field variable which can be associated with every event except a small set can equally well be associated with the small set through the Hodge star duality transformation

$$\Xi \to *\Xi$$

The invariants can now be written as expressions combining the components and their duals which satisfy our ideas of locality.

Having constructed such a satisfying model which seems to unify space-time and matter, we might well feel encouraged to study its dynamical behavior with some sense of optimism. However, it is well known that the gamma matrices which generate the Clifford algebra have a representation in matrices of size $D \times D$, where

$$D = 2^{N/2}$$

Because of the Grassmann variables, these can be taken as supermatrices. Since the dimension of the algebra is the same as the dimension of the matrices as a vector space, it follows that there is an isomorphism between the Clifford algebra and the algebra of supermatrices over complex numbers. The invariants we have used are merely the trace of these matrices to the nth power and it follows that the model we have described is mathematically equivalent to a supermatrix model. Such models are not likely to be rich enough to provide a complete description of physics.

Despite this, the model has interesting properties and we will go on to find that modifications to the model can make it more promising. It is also worth noting the possibility of relationships with other applications of Clifford algebras to models of space-time physics (Finkelstein, 1982; Smith, 1994).

9. EVENT-SYMMETRIC STRING THEORY

Despite the enormous number of papers written on superstring theory and the rich mathematics discovered in the course of that research, physicists still appear to be far from understanding its origins. It is generally believed that string theory has a huge hidden symmetry which is restored at very high energies (Gross, 1988). If the nature of that symmetry could be understood, then it might be possible to construct a fundamental formulation of string theory which would allow its nonperturbative phenomenology to be studied.

A result of great significance here is that in string theory it is possible to make smooth transitions between topologically distinct space-time backgrounds (Aspinwall *et al.*, 1994). As I have already argued, the combined requirements of space-time symmetry and topology change seem to force us to accept the principle of event-symmetric space-time.

This is sufficient justification to seek an event-symmetric model of string field theory. That is not an easy task since there is no completely satisfactory formulation of continuum string theory which might be discretized in some event-symmetric fashion. One clue must be matrix models which are equivalent to $c = 0$ string theories and which we can interpret as event-symmetric. We should also take into account the Clifford algebra model which we saw as a model of fermions but which also included supersymmetry.

If we could find a suitable description of string symmetry, then the job would be at least half complete. For mathematicians, classifying symmetries has been a priority problem throughout the 20th century. Most promising for our purposes must be the various forms of Kac–Moody algebras and quantum groups which are related to conformal field theory (see, e.g., Pressley and Segal, 1988; Fuchs, 1992). Kaku tried to formulate symmetry for string theory in terms of Lie algebras described on topological strings (Kaku, 1988, 1990). Other new forms of symmetry have been found in string theory such as W_∞-algebras (e.g., Shen, 1992; Bouwknegt and Schoutens, 1993) and it is known that string theory compacted onto a 26-dimensional torus possesses a symmetry known as the Fake Monster Lie algebra (see, e.g., Gebert, 1993). Despite all these discoveries, there are large gaps in the understanding of infinite-dimensional symmetry algebras and nothing is known which can include all the supposed symmetries of string theory while at the same time unifying space-time symmetries with internal gauge symmetries and explaining its remarkable dualities (e.g., Hull and Townsend, 1995).

In an event-symmetric space-time a string is most easily represented by a loop connecting a cycle of space-time events and is therefore an object made of discrete points. This may seem unnatural since string theory is normally regarded as a theory of continuous strings. However, it is possible that strings are topological in nature and could be exactly described as discrete strings with a finite spacing between events (Klebanov and Susskind, 1988; Thorn, 1991; Kostov, 1995). The topological form will most likely become apparent through a q-deformation in which the partons of the discrete strings take on fractional statistics.

In a number of preprints (Gibbs, 1994b,c, 1995a) I have tried to construct Lie algebras based on such discrete loops in analogy with Kaku's string groups. Although this work produced many positive results, it turned out to be flawed since the Lie superalgebras I constructed for closed loops do not satisfy the graded Jacobi identity in all cases (Borcherds, 1995). The result of correcting the anomaly is a tidier formulation which I believe has much more promise for the possibility of generalization and deformation. It will be presented in its most basic form here for closed strings.

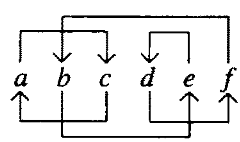
Let E be a set of N space-time events and let $V = span(E)$ be the N-dimensional vector space spanned by those events. Then define $T = Tensor(V)$ to be the free associative algebra with unit generated over V. The components of T form an infinite family of tensors over V with one representative of each rank,

$$\Phi = \{\varphi, \varphi_a, \varphi_{ab}, \varphi_{abc}, \ldots\}$$
$$\Phi^1\Phi^2 = \{\varphi^1\varphi^2, \varphi^1\varphi^2_a + \varphi^1_a\varphi^2, \varphi^1\varphi^2_{ab} + \varphi^1_a\varphi^2_b + \varphi^1_{ab}\varphi^2, \ldots\}$$

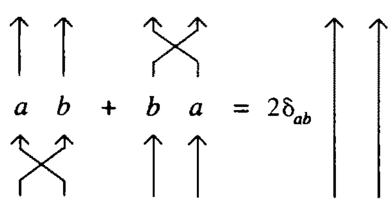
The basis of this algebra already has a geometric interpretation as open strings passing through a sequence of events with arbitrary finite length. Multiplication of these strings consists merely in joining the end of the first to the start of the second. We can denote this as follows:

$$\Phi = \varphi + \sum_a \varphi_a a + \sum_{a,b} \varphi_{ab} ab + \sum_{a,b,c} \varphi_{abc} abc + \ldots$$

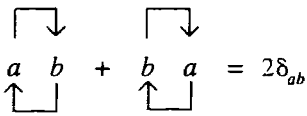
We now construct a new algebra by adding an extra connectivity structure to each string consisting of arrows joining events. There must be exactly one arrow going into each string and one leading out. This structure defines a permutation of the string events so there are exactly $K!$ ways of adding such a structure to a string of length K,



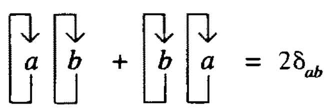
These objects now form the basis of a new algebra with associative multiplication consisting of joining the strings together as before, while preserving the connections. Finally the algebra is reduced modulo commutation relations between events in strings which are defined schematically as follows:



These are partial relations which can be embedded into complete relations. Closed loops which include no events are identified with unity. For example, the lines can be joined to give



This example shows the cyclic relation on a loop of two events. The arrows can be joined differently to give another relation,



which is the anticommutation relation for loops of single events.

By applying these relations repeatedly, it is possible to reorder the events in any string so that the strings are separated into products of ordered cycles. Therefore we can define a more convenient notation in which an ordered cycle is indicated as follows:

$$(ab\ldots c) = a \rightarrow b \rightarrow \ldots \rightarrow c$$

We can generate cyclic relations for loops of any length such as

$$(ab) = -(ba) + 2\delta_{ab}$$

$$(abc) = (cab) + 2\delta_{bc}(a) - 2\delta_{ac}(b)$$

$$(abcd) = -(dabc) + 2\delta_{cd}(ab) - 2\delta_{bd}(a)(c) + 2\delta_{ad}(bc)$$

and graded commutation relations such as

$$(a)(b) + (b)(a) = 2\delta_{ab}$$

$$(ab)(c) - (c)(ab) = 2\delta_{bc}(a) - 2\delta_{ac}(b)$$

Clearly the algebra has a Z_2 grading given by the parity of the length of string and it is therefore possible to construct an infinite-dimensional Lie superalgebra using the graded commutator.

The length-one cycles are the generators of a Clifford algebra and there is also a homomorphism from the full algebra onto a Clifford algebra defined by removing the loop structure from the strings.

The physical interpretation is that this algebra describes the symmetry of a discrete superstring formed from loops of fermionic partons in event-symmetric space-time. Mathematically it appears to be an entirely new type of symmetry which is likely to have generalizations and deformations that could be of some significance.

10. CONCLUSIONS

I have introduced the principle of event-symmetric space-time and argued for its validity despite its unlikely-seeming consequences. In event-symmetric models the nature of space-time, including its topological structure, is dynamically determined. A physical consequence is that at very high temperatures space-time may change dimension or even evaporate, losing all sense of causality and locality.

In a series of toy models I have tried to gain a feel for what a correct event-symmetric theory should look like and behave like. This has led to algebraic models with high degrees of symmetry. The most advanced models are event-symmetric discrete string theories.

To finish the work on event-symmetric string theory it will probably be necessary to deform the string algebras described here. It is probably necessary to model a string as a loop of particles with fractional statistics rather than fermions. Such a deformation might be possible if the loops are replaced with knots.

To complete the theory it will also be necessary to define the dynamics of the system and discover a correspondence with recognized space-time physics. There is still a long way to go.

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